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ZEROES OF HOLOMORPHIC VECTOR FIELDS AND GROTHENDIECK DUALITY THEORY

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ABSTRACT. The holomorphic fixed point formula of Atiyah and Bott is discussed in terms of Grothendieck's theory of duality. The treatment is valid for an endomorphism of a compact complex-analytic manifold with arbitrary isolated fixed points. An expression for the fixed point indices is then derived for the case where the endomorphism belongs to the additive group generated by a holomorphic vector field with isolated zeroes. An application and some examples are given. Two generalisations of these results are also proved. The first deals with holomorphic vector bundles having sufficient homogeneity properties with respect to the action of the additive group on the base manifold, and the second with additive group actions on algebraic varieties.

1. **Introduction.** This paper is devoted to a development of previous results [16] concerning the calculation of the local fixed point contributions in the holomorphic fixed point formula of [2], in the case where the endomorphism of the manifold has possibly nontransversal fixed points and belongs to the one-parameter group generated by a holomorphic vector field with isolated zeroes. In this situation it was shown in [16] that the fixed point contributions are entire meromorphic functions of a certain form, and are given by expressions involving the Grothendieck residue, and Todd polynomials in the 'Chern classes' associated to a zero of a holomorphic vector field [3].

The main innovation of this paper is the use, in the context of Grothendieck's theory of duality [1], [5] of a standard iterative procedure often employed in proving the existence of integral curves of a vector field. The results obtained immediately imply Theorem 1 of [16] but the method of proof which will be used is superior for several reasons. Firstly, the basic theorem of [16] is that two local cohomology elements (see below) have the same image under the Grothendieck residue. Here it is proved that they are actually the same element–a considerably stronger result. Secondly, the present method allows generalisations in two directions; to more general

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coefficients for the cohomology groups occurring in the fixed point formula (§5), and to algebraic varieties over more general fields (§7).

Since there appears to be little in the literature concerning the application of Grothendieck duality theory to the holomorphic fixed point formula, a brief discussion of this topic has also been included.

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2. **Definitions and results.** Let M be a complex analytic manifold, of complex dimension n, and X a holomorphic vector field on M, i.e. a holomorphic section of the holomorphic tangent bundle TM of M. For $t \in \mathbf{C}$ sufficiently small, let $z \mapsto f(z, t)$ be the one-parameter family of complex analytic endomorphisms of M generated by X (see §4). Let $z_1, \ldots, z_n$ be local holomorphic coordinates centred at $p \in M$, and if $\mathcal{O}_M$ is the sheaf of germs of holomorphic functions on M, we let $\mathcal{O}_p$ be the local ring at p. The corresponding local ring at $(p, 0) \in M \times \mathbf{C}$ will be denoted by $\mathcal{O}_p'$, and $\mathcal{O}_p$ may be identified with a subring of $\mathcal{O}_p'$ via the projection $(z, t) \mapsto z$. Similarly an element of $\mathcal{O}_p'$ may be regarded as a family of elements of $\mathcal{O}_p$ parametrized by t, for t sufficiently small.

Then X may be expressed in a neighborhood of p by:

$$X(z) = \sum_{i=1}^{n} a_i(z)\partial/\partial z_i \tag{2.1}$$

with each a_i holomorphic. Since $f(z, 0) = z$, it makes sense to consider the coordinates $f_i(z, t)$ of f for small t, and z near p. In the following, we make the abbreviation $z_i^t = f_i(z, t)$, and holomorphic functions will be identified with their germs whenever this is convenient.

THEOREM 1. *Let $\mathcal{J}_p(z - z^t) \subset \mathcal{O}_p'$ be the ideal generated by the germs of the functions $z_i - z_i^t$ for $1 \leqslant i \leqslant n$, and let $\mathcal{J}_p(a) \subset \mathcal{O}_p$ be the ideal generated by the germs of the $a_i(z)$. Then $\mathcal{J}_p(z - z^t) = t.\mathcal{J}_p(a)$.*

Before stating the next theorem we introduce the following notation. Let $A(z)$ be the $n \times n$ matrix over the ring $\mathcal{O}_p$ given by the partial derivatives of the a_i:

$$A_{ij}(z) = \partial a_i/\partial z_j(z). \tag{2.2}$$

We note that

$$\tau = \sum_{k=0}^{\infty} t^k A^k/(k+1)! \tag{2.3}$$

is an invertible element in the ring of $n \times n$ matrices over $\mathcal{O}_p'$. Formally we may write

$$\tau^{-1} = tA/(I - e^{tA}).$$

Then we can define an element $T(z, t) \in \mathcal{O}_p'$ by:

$$T(z, t) = \det\big(tA / (I - e^{tA})\big).$$

Theorem 1 is of course trivial at any point where the vector field does not vanish. In order to apply the theorem, we again make the restriction that p is an *isolated* zero of X. We then obtain the following:

THEOREM 2. *Let Ω^k be the sheaf of germs of holomorphic k-forms on M, and let dz be the element $dz_1 \wedge \cdots \wedge dz_n$ of Ω_p^n. Then, for $t \in \mathbf{C} - \{0\}$ sufficiently small, the following equality holds in* $\operatorname{Ext}^n_{\mathcal{O}_p}(\mathcal{O}_p/\mathcal{I}_p(a), \Omega_p^n)$ *(see below for notation)*:

$$\text{(2.4)} \qquad \left[\frac{dz}{z_1 - z_1^t, \ldots, z_n - z_n^t}\right] = \left[\frac{t^{-n} \times T(z, t)dz}{a_1, \ldots, a_n}\right].$$

The main interest of this theorem lies in its connection with the fixed point theorem of Atiyah-Bott for the sheaf $\mathcal{O}_M$ on a compact complex-analytic M. Then the first element of (2.4) arises naturally at an isolated fixed point of a holomorphic map $f: M \to M$, and Theorem 2 gives an alternative description of this element in the case where f is the flow associated to a holomorphic vector field with isolated zeroes. In fact, this theorem is a strengthening of the result obtained in [16], where it was shown only that the two sides of (2.4) have the same image under the Grothendieck residue map (see §3). Note that the right-hand side of (2.4) has a much more explicit dependence on t than does the left side. This is the fact that was exploited in [16] in order to investigate the properties of the meromorphic function in t obtained by applying the residue map.

Theorem 2 is an easy consequence of Theorem 1 and the fundamental local isomorphism of [1, I.4] or [9, III.7]. In the present case, this isomorphism may be described as follows. For further details, see [loc. cit.].

Let $(Z, \mathcal{O}_Z)$ be an analytic subspace of the complex analytic manifold $(M, \mathcal{O}_M)$ which is locally a complete intersection of codimension r. This means that each point of Z has a neighborhood U in M on which there are holomorphic functions $g_1, \ldots, g_r$ with the property that $\mathcal{O}_Z$ is the quotient of $\mathcal{O}_M$ by the ideal $\mathcal{J}$ generated by the g_i, and at each point $z \in Z \cap U$ the germs $g_{i,z}$ of the g_i form a regular $\mathcal{O}_z$-sequence (see [8, 0.15] or [1, III.3.1]).

The two examples of local complete intersections which will be of interest here are:

(1) Any nonsingular complex analytic submanifold of M, with its standard structure sheaf.

(2) If $g_1, \ldots, g_n$ are analytic functions defined in a neighborhood of $p \in M$, with an isolated common zero at p, then the g_i define the structure of

a (not necessarily reduced) analytic subspace on $\{p\}$, which is a local complete intersection in M.

Now, if $\mathcal{F}$ is a locally free $\mathcal{O}_M$-Module, and $\mathcal{J}$ the ideal of $\mathcal{O}_M$ such that $\mathcal{O}_Z = \mathcal{O}_M/\mathcal{J}$, the Koszul complex defined by the functions g_i gives, over U, an isomorphism:

$$\mathcal{H}om_{\mathcal{O}_Z}\left(\bigwedge^r(\mathcal{J}/\mathcal{J}^2), \mathcal{F}/\mathcal{J}\mathcal{F}\right) \xrightarrow{\sim} \mathcal{E}xt^r_{\mathcal{O}_M}(\mathcal{O}_Z, \mathcal{F})$$

where the notation is that of [1]. This isomorphism is independent of the choice of the g_i, and so defines a canonical global isomorphism.

Also $\bigwedge^r(\mathcal{J}/\mathcal{J}^2)$ is an invertible $\mathcal{O}_Z$-Module, generated locally by the section $[g_1] \wedge \cdots \wedge [g_r]$, where the square brackets indicate the class mod $\mathcal{J}^2$. Thus if s is any section of $\mathcal{F}$ over U, and (s) its class mod $\mathcal{J}\mathcal{F}$, the homomorphism defined by:

$$[g_1] \wedge \cdots \wedge [g_r] \mapsto (s)$$

gives a section of $\mathcal{E}xt^r_{\mathcal{O}_M}(\mathcal{O}_Z, \mathcal{F})$ over U which will be denoted by

$$\left[\frac{s}{g_1, \ldots, g_r}\right].$$

In order to motivate some of these constructions, the next paragraph will outline a proof of the relevant fixed point formula for isolated but not necessarily transversal fixed points, using the fundamental local isomorphism and the techniques of local cohomology [7], together with the closely related local and global Ext functors of [1], [5], [6]. This approach is apparently well known, but since it differs from that of other published versions ([18], [20], [21]) it seems worthwhile to repeat the main points in order to place some of the concepts used in the preceding paragraph in their natural setting. A similar approach has been used recently by D. Toledo and Y.L.L. Tong in the context of a holomorphic Lefschetz formula for nonisolated fixed points.

In §5 the above results are generalised to calculate the fixed-point indices for the case of coefficients in any bundle which satisfies certain homogeneity conditions with respect to the action of the additive broup on the base manifold. The main result obtained is similar to Theorem 2 above, the formula for the fixed-point index being modified by a factor analogous to the Chern character in 'Chern classes' associated to the bundle at a zero of the vector field.

The next section contains some examples, and the final section treats the case of a smooth algebraic variety defined over an algebraically closed field of characteristic zero which is acted upon by the additive group of the field.

3. **A review of the holomorphic fixed-point formula.** Let M, $\mathcal{O}_M$, Ω^k be as in §2, with the additional restriction that M be compact, and suppose that $\mathcal{E}$ is a locally free $\mathcal{O}_M$-Module of rank m, with dual $\mathcal{E}^*$. We define

$$\mathcal{E}' = \Omega^n \otimes_{\mathcal{O}_M} \mathcal{E}^*$$

and there is a natural pairing:

$$\mathrm{Tr}\colon \mathcal{E}' \otimes_{\mathcal{O}_M} \mathcal{E} \to \Omega^n.$$

We may also form the product $M \times M$, with projection maps p_1, p_2 and diagonal map Δ. The locally free sheaf $p_1^*\mathcal{E}' \otimes_{\mathcal{O}_{M\times M}} p_2^*\mathcal{E}$ will be denoted by $\mathcal{E}' \boxtimes \mathcal{E}$.

Let $\mathcal{J}$ be the ideal of $\mathcal{O}_{M\times M}$ consisting of germs vanishing on the diagonal, so that there is an exact sequence:

$$0 \to \mathcal{J} \to \mathcal{O}_{M\times M} \to \Delta_*\mathcal{O}_M \to 0. \tag{3.1}$$

Then, if $z_1, \ldots, z_n$ are local coordinates centred at $p \in M$, there are corresponding local coordinates z_i, ζ_i centred at $(p, p) \in M \times M$, given by $p_1^*z_i$ and $p_2^*z_i$ respectively. Also, if generating sections $s_1, \ldots, s_m$ are chosen for $\mathcal{E}$ in a neighborhood of p, there are corresponding dual sections s_i^* of $\mathcal{E}^*$. Moreover, the functions $z_i - \zeta_i$, for $1 < i < n$, form a regular sequence of parameters generating the ideal $\mathcal{J}$ around p, and there is a canonical isomorphism, independent of any choice of coordinates:

$$\mathcal{J}/\mathcal{J}^2 \xrightarrow{\sim} p_1^*\Omega^1/\mathcal{J}.p_1^*\Omega^1$$

given locally by $[z_i - \zeta_i] \to (dz_i)$. This then gives a canonical isomorphism:

$$\mathcal{H}om_{\Delta_*\mathcal{O}_M}\big(\wedge^n(\mathcal{J}/\mathcal{J}^2), \Delta_*(\mathcal{E}' \otimes_{\mathcal{O}_M} \mathcal{E})\big) \xrightarrow{\sim} \Delta_*\mathcal{E}nd_{\mathcal{O}_M}(\Omega^n \otimes_{\mathcal{O}_M} \mathcal{E}).$$

The 'identity' section of this latter sheaf then defines, via the fundamental local isomorphism, a global section of the sheaf $\mathcal{E}xt^n_{\mathcal{O}_{M\times M}}(\Delta_*\mathcal{O}_M, \mathcal{E}' \boxtimes \mathcal{E})$. In the notation of §2 this section may be written locally as

$$\left[\frac{dz \otimes \sum_{i=1}^m s_i^*(z) \otimes s_i(\zeta)}{z_1 - \zeta_1, \ldots, z_n - \zeta_n}\right]. \tag{3.2}$$

In this situation the sheaves $\mathcal{E}xt^k_{\mathcal{O}_{M\times M}}$ vanish for $k < n$ [1, III.3.8], as do the corresponding local cohomology sheaves $\mathcal{H}^k_\Delta(\mathcal{E}' \boxtimes \mathcal{E})$ [19, Theorem 3.3]. Therefore both the spectral sequence of local and global Ext [7, p. 31] and the local cohomology spectral sequence [7, p. 12] degenerate, and the natural transformation of functors:

$$\mathrm{Hom}\big(M \times M, \Delta_*\mathcal{O}_M, \cdot\big) \to \Gamma_\Delta(M \times M, \cdot)$$

leads to a commutative diagram [7, p. 35] in which the horizontal maps are the edge-homomorphisms of the above spectral sequences, and so in this case are isomorphisms:

$$\begin{array}{ccc} H^0(M \times M, \mathcal{E}xt^n(\Delta_*\mathcal{O}_M, \mathcal{E}' \boxtimes \mathcal{E})) & \xleftarrow{\sim} & \mathrm{Ext}^n(M \times M, \Delta_*\mathcal{O}_M, \mathcal{E}' \boxtimes \mathcal{E}) \\ \downarrow & & \downarrow \\ H^0(M \times M, \mathcal{H}^n_\Delta(\mathcal{E}' \boxtimes \mathcal{E})) & \xleftarrow{\sim} & H^n_\Delta(M \times M, \mathcal{E}' \boxtimes \mathcal{E}) \end{array} \tag{3.3}$$

Thus the global section (3.2) gives an element δ_Δ of $H^n_\Delta(M \times M, \mathcal{E}' \boxtimes \mathcal{E})$, and hence a cohomology class in $H^n(M \times M, \mathcal{E}' \boxtimes \mathcal{E})$, which we denote by δ.

PROPOSITION 1. *Under the isomorphisms*

$$H^n(M \times M, \mathcal{E}' \boxtimes \mathcal{E}) \xrightarrow{\sim} \sum_{k=0}^{n} H^{n-k}(M, \mathcal{E}') \otimes H^k(M, \mathcal{E})$$
$$\xrightarrow{\sim} \sum_{k=1}^{m} \operatorname{End}_{\mathbf{C}} H^k(M, \mathcal{E})$$

given by the Künneth formula and Serre duality, the cohomology class δ *corresponds to*

$$\sum_{k=0}^{n} (-1)^k \times \text{ identity on } H^k(M, \mathcal{E}).$$

PROOF. The proposition is equivalent to the fact that δ_Δ is represented by the current of type $(n, 0)$ with coefficients in $\mathcal{E}' \boxtimes \mathcal{E}$ which, for u a $(0, n)$-form on $M \times M$ with coefficients in $\mathcal{E} \boxtimes \mathcal{E}'$, is defined by:

$$u \mapsto \int_M \operatorname{Tr} \cdot \Delta^*(u).$$

In view of (3.3) it is sufficient to prove this locally, in which case one can use an analogue of the Dolbeault homomorphism, as in [10]. Note that the required current is essentially $\bar{\partial} k$, where k is the Bochner-Martinelli kernel:

$$k(z, \zeta) = C_n \sum_{i,j} (-1)^{i+1} \frac{\bar{z}_i - \bar{\zeta}_i}{|z - \zeta|^{2n}} d(\bar{z}_1 - \bar{\zeta}_1) \wedge \cdots \wedge \widehat{d(\bar{z}_i - \bar{\zeta}_i)} \wedge \cdots$$
$$\wedge d(\bar{z}_n - \bar{\zeta}_n) \wedge dz_1 \wedge \cdots \wedge dz_n \otimes s_j^*(z) \otimes s_j(\zeta)$$

where

$$C_n = (-1)^{n(n-1)/2} \times \frac{(n-1)!}{(2\pi i)^n} \quad \text{and} \quad |z - \zeta|^2 = \sum_{i=1}^{n} |z_i - \zeta_i|^2.$$

The Bochner-Martinelli kernel has already been used in this context in [20], [21]. The analogy with [10] is easily made once one observes the relationship [8, III.1.2] between the construction of the functors Ext via Koszul complexes, and the Čech cohomology of the complement of the diagonal, in which the section (3.2) corresponds to the *Cauchy kernel* of [10].

Now let f: $M \to M$ be a holomorphic map with isolated fixed points. We also assume the existence of an $\mathcal{O}_M$-module homomorphism

$$\varphi : f^*\mathcal{E} \to \mathcal{E}.$$

This corresponds to the geometric endomorphism of [2]. In the case where $\mathcal{E} = \Omega^k$, φ may be taken to be the map induced by the kth exterior power of the differential of f.

Then there are induced endomorphisms $H^k(f, \varphi)$ of the cohomology groups given by the composition

$$H^k(M, \mathcal{E}) \xrightarrow{f^*} H^k(M, f^*\mathcal{E}) \xrightarrow{\varphi} H^k(M, \mathcal{E})$$

and it is elementary to check, using Proposition 1 and the definition of the Serre duality pairing, that

$$\int_M \mathrm{Tr} \cdot 1 \otimes \varphi \cdot \Gamma^*(\delta) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}_{\mathbf{C}}\, H^k(f, \varphi)$$

where Γ is the graph morphism, $\Gamma(x) = (x, f(x))$.

However, the fact that δ has support on the diagonal, being the image of $\delta_\Delta \in H^n_\Delta(M \times M, \mathcal{E}' \boxtimes \mathcal{E})$ implies that $\Gamma^*(\delta)$ has support on the fixed point set F of f, and is the image of an element $\delta_F \in H^n_F(M, \mathcal{E}' \otimes_{\mathcal{O}_M} f^*\mathcal{E})$. Let

$$0 \to \mathcal{I} \to \mathcal{O}_M \to \mathcal{O}_F \to 0 \tag{3.4}$$

be the sequence obtained by applying Γ^* to (3.1). Using the fact that the fixed points are isolated, one shows that this sequence is exact, and that $\mathcal{I}$ is generated in a neighborhood of each fixed point by the regular sequence of parameters $z_i - f_i(z)$, for $1 \leqslant i \leqslant n$. It is then simply a matter of checking the necessary compatibilities to conclude that the section of

$$\mathcal{E}xt^n_{\mathcal{O}_M}(\mathcal{O}_F, \mathcal{E}' \otimes_{\mathcal{O}_M} f^*\mathcal{E})$$

given in a neighborhood of each fixed point by

$$\left[\frac{dz \otimes \sum_i s_i^*(z) \otimes s_i(f(z))}{z_1 - f_1(z), \ldots, z_n - f_n(z)} \right] \tag{3.5}$$

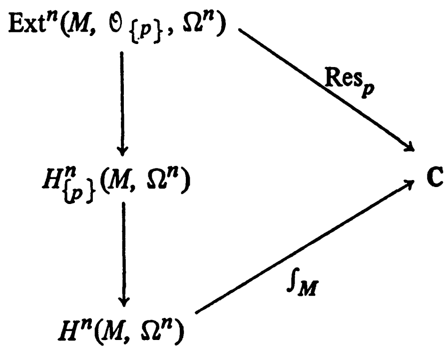
defines an element of $\mathrm{Ext}^n(M, \mathcal{O}_F, \mathcal{E}' \otimes_{\mathcal{O}_M} f^*\mathcal{E})$ which corresponds to the local cohomology element δ_F. Note that (3.5) is independent of choices of coordinates and generating sections since it is simply the restriction of (3.2) to the graph. By applying $\mathrm{Tr} \cdot 1 \otimes \varphi$ we obtain an element of $\mathrm{Ext}^n(M, \mathcal{O}_F, \Omega^n)$ which is the direct sum of elements α_p for $p \in F$, with

$$\alpha_p = \left[\frac{\mathrm{trace}\, \varphi(z) dz_1 \wedge \cdots \wedge dz_n}{z_1 - f_1(z), \ldots, z_n - f_n(z)} \right]$$

where $\mathrm{trace}\, \varphi(z) = \sum_i \langle s_i^*(z), \varphi(s_i(f(z)))\rangle$. In case $\mathcal{E} = \Omega^k$ and φ is induced by the differential of f, then trace φ is simply trace $\bigwedge^k(\partial f_i / \partial z_j(z))$.

At this point we may introduce the Grothendieck residue map of [9]. Precisely, if $p \in M$ and $\mathcal{O}_{\{p\}}$ is an $\mathcal{O}_M$-Module with support $\{p\}$, such that $(p, \mathcal{O}_{\{p\}})$ is a local complete intersection in M, the residue is the map Res_p

which makes the following diagram commute:



where the upper vertical map is induced by the natural transformation of functors

$$\mathrm{Hom}(M, \mathcal{O}_{\{p\}}, \cdot) \to \Gamma_{\{p\}}(M, \cdot).$$

However, Res_p may be defined by purely algebraic means, as in [9], and in our case may be evaluated by the algorithm described in [3]. The fixed-point formula now follows immediately from the commutativity of the above diagram:

$$\sum_{k=0}^{n} (-1)^k \mathrm{trace}_{\mathbf{C}} H^k(f, \varphi) = \sum_{p \in F} \mathrm{Res}_p(\alpha_p).$$

We may also note that around each fixed point p we can restrict the Bochner-Martinelli kernel to the graph. This defines the $\bar{\partial}$-cohomology class of a current of type (n, n) with support at p, and it again reduces the checking compatibilities to show that the residue $\mathrm{Res}_p(\alpha_p)$ is given by evaluating this current on 1. Thus we recover the integral formula of [20], [21]:

$$\mathrm{Res}_p(\alpha_p) = \int_S \mathrm{Tr} \cdot 1 \otimes \varphi \cdot \Gamma^*(k)$$

where S is any small, smooth $2n - 1$ sphere about p. See also [17] in this context. This concludes our brief discussion of the holomorphic fixed-point formula.

4. Proofs of Theorems 1 and 2. With M, X as in the Introduction, there is an open neighborhood D of $M \times \{0\}$ in $M \times \mathbf{C}$ and a one-parameter family of holomorphic endomorphisms of M generated by X, called the *flow* associated to X, f: $D \to M$. In fact, if $t = uv$ with u in the unit circle in the complex plane and v real, then $f(z, t) = f_u(z, v)$, where f_u is the one-parameter family generated by the *real* vector field $uX + \overline{uX}$ on M. Moreover if M is compact, D may be taken to be the whole of $M \times \mathbf{C}$.

A method often used to demonstrate the existence of such a flow is the iterative procedure found, for example, in [14]. It is shown below how this device can be adapted to the present situation in order to prove the required results.

Let U be an open neighborhood of $p \in M$, on which the vector field has the form (2.1). Then, for $t \in \mathbf{C}$ sufficiently small, the resulting flow f is characterised by the conditions, for $1 \leqslant i \leqslant n$,

$$\partial f_i / \partial t(z, t) = a_i(f(z, t)) \quad \text{and} \quad f_i(z, 0) = z_i.$$

Let $V \subset U$ be open and let W be a disc centred on the origin in $\mathbf{C}$. Then if V and W are chosen to be sufficiently small [14, IV.1] we may inductively define functions $f^{(n)}\colon V \times W \to U$ by:

$$f^{(0)}(z, t) = z$$

and for $m > 0$:

$$f^{(m)}(z, t) = z + \int_0^t a\big(f^{(m-1)}(z, s)\big)\, ds$$

where $a\colon U \to \mathbf{C}^n$ is the function $a(z) = (a_1(z), \dots, a_n(z))$. Note that each $f^{(m)}$ is holomorphic, so that the integral may be taken along any smooth path from 0 to t in W.

Again, as in [14, IV.1], if V and W are small enough, the sequence $f^{(m)}$ converges uniformly on $V \times W$ and the limit f is the local flow associated to the vector field. This also shows that f is holomorphic, being the uniform limit of holomorphic mappings [12, 2.2.4].

The elements of $\mathcal{O}_p'$ can be expressed as convergent power series, and $\mathcal{O}_p'$ will be given the topology of simple convergence of the coefficients [12, p. 155]. Thus $f^{(m)} \to f$ if each coefficient in the power series expansion of $f^{(m)}$ converges to the corresponding coefficient in the expansion of f.

We first prove the following lemma.

Lemma 1. *Let A be the matrix of partial derivatives* (2.1). *Then, in matrix notation, the following holds in $\mathcal{O}_p'^n$ for $m > 0$:*

$$f^{(m)}(z, t) - z \equiv \sum_{k=1}^{m} \frac{t^k}{k!} \cdot A^{k-1} \cdot a(z) \mod\big(t.\mathcal{I}_p(a)\big)^2.\ \mathcal{O}_p'^n. \tag{4.1}$$

Proof. The proof is a simple induction. The result is trivially true for $m = 1$, and if true for $m = N$ then, using the fact that ideals are closed in $\mathcal{O}_p'$ in the topology of simple convergence [12, 6.3.5],

$$\begin{aligned} a\big(f^{(N)}(z, t)\big) &= a\big(z + f^{(N)}(z, t) - z\big) \\ &\equiv a(z) + A(z) \cdot \big(f^{(N)}(z, t) - z\big) \mod\big(\mathcal{I}_p(z - f^{(N)})\big)^2.\ \mathcal{O}_p'^n. \end{aligned}$$

The result for $m = N + 1$ then follows from the case $m = N$ and the definition of $f^{(N+1)}$.

We are now in a position to prove the theorem. First note that since the functions $f_i^{(m)}$ converge uniformly in a neighborhood of $(p, 0)$ in $V \times W$, they also converge in $\mathcal{O}_p'$ by the Cauchy inequalities [12, 2.2.7]. The same applies to the partial sums on the right-hand side of (4.1). Then, again using the fact that ideals of $\mathcal{O}_p'$ are closed, we can let $m \to \infty$ in (4.1) and conclude, with τ defined by (2.3), that

$$z - f(z, t) \equiv \tau(z, t) \cdot ta(z) \mod (t.\mathcal{I}_p(a))^2 \mathcal{O}_p'^n. \tag{4.2}$$

Because τ is invertible, this means that

$$t.\mathcal{I}_p(a) = \mathcal{I}_p(z - z') + (t.\mathcal{I}_p(a))^2$$

and the theorem follows from Nakayama's lemma.

In order to prove Theorem 2, first observe that

$$z - z' \equiv ((I - e^{tA})/tA) \cdot ta \mod \mathcal{I}_p^2.\mathcal{O}_p'^n$$

where $\mathcal{I}_p$-$t.\mathcal{I}_p(a)$. Theorem 2 now follows from the definitions.

5. **A generalisation.** We recall that the fixed-point theorem can be proved for cohomology with coefficients in the sheaf of germs of sections of any holomorphic vector bundle E for which there exists a suitable geometrical endomorphism (see §3). In the present situations we require the existence of a family of geometrical endomorphisms of E corresponding to the group of endomorphisms of M induced by the vector field. Sufficient conditions for the existence of such a family are obtained in the situation discussed below, which will be formulated in the context of an arbitrary group G acting (on the left) on the complex-analytic manifold M by holomorphic transformations.

In this case M will be said to be a *holomorphic G-space*, and for $g \in G$ the corresponding endomorphism of M will also be denoted by g.

First recall the following definition.

DEFINITION 1. The holomorphic vector bundle E is said to be a *holomorphic G-bundle* if:

(i) E is a holomorphic G-space.

(ii) The projection $E \to M$ commutes with the action of G.

(iii) For $g \in G$ and $x \in M$ the map $E_x \to E_{g(x)}$ is complex linear.

For example, any combination of tensor or exterior powers of the holomorphic tangent bundle of M is a holomorphic G-bundle. Other examples occur on the homogeneous spaces of Lie groups [2].

REMARK. If E is a holomorphic G-bundle then so is the holomorphic dual E^*. To see this, for $x \in M$ let g_x^{-1}: $E_{g(x)} \to E_x$ be the map induced by the action of g^{-1} on E. Then the adjoint maps $E_x^* \to E_{g(x)}^*$ give an action of g on

E^* which makes E^* into a holomorphic G-bundle. Thus tensor and exterior powers of the holomorphic cotangent bundle are holomorphic G-bundles.

The point of this definition is that for G-vector bundles there always exist geometrical endomorphisms compatible with the action of G; by Definition 1 and the universal property of the pull-back there exists a holomorphic bundle map over M for all $g \in G$,

$$E \to (g^{-1})^* E.$$

By taking the pull-back relative to the map g: $M \to M$ we obtain the required endomorphism, which will be denoted by

$$\varphi_g : g^* E \to E.$$

Note that if E is a G-vector bundle this gives a (right) representation of G on the space $C^\infty(M, E)$ of smooth global sections of E, which will be written $s \mapsto s^g$ for $s \in C^\infty(M, E)$, where $s^g(x) = \varphi_g(s(g(x)))$. It is easy to check that this defines a representation.

Let $x \in M$ be a fixed point for the action of G, i.e. $g(x) = x$ for all $g \in G$. Then if $\mathcal{E}_x$ is the space of germs of holomorphic sections of E at x, there is a (right) representation of G on this space which will again be denoted by $s \mapsto s^g$, where as before $s^g = \varphi_g \circ s \circ g$.

In the case where E is the trivial line bundle the resulting representation on the local ring $\mathcal{O}_x$ is simply $f \mapsto f^g$ where $f^g(x) = f(g(x))$. This is therefore compatible with the notation of §2.

We now return to the case where $G = \mathbf{C}^+$ and the $\mathbf{C}^+$-action on M is induced by a holomorphic vector field X with isolated zeroes. Let p be a zero of the vector field and let $s_1, \dots, s_m$ be holomorphic sections generating the $\mathbf{C}^+$-vector bundle E in a neighbourhood of p, with corresponding dual sections $s_1^*, \dots, s_m^*$ of E^*. Recall that the expression which enters into the fixed-point index at p is the class in $\mathcal{O}_p'/\mathcal{I}_p$ of

$$\operatorname{trace} \varphi_t(z) = \sum_{i=1}^{m} \langle s_i^*, s_i^t \rangle.$$

As before, it is not clear how this expression depends on t. This dependance can be clarified as follows. We identity sections of E with their germs at p whenever this is convenient.

For $1 \leqslant i \leqslant m$ we may differentiate s_i^t in $\mathcal{E}_p$ with respect to t and obtain

$$\partial s_i^t / \partial t\big|_{t=0} = \sum_{j=1}^{n} L_{ij} s_j$$

with $L_{ij} \in \mathcal{O}_p$. Since $s \mapsto s^t$ is a representation of $\mathbf{C}^+$, this implies that

$$\partial s_i^t / \partial t\big|_{t=u} = \sum_{j=1}^{n} L_{ij}^u s_j^u.$$

This is a system of first-order differential equations for the s_j^t which may be integrated in precisely the same way as the vector field of §4. We write $s = (s_1, \ldots, s_m)$ and, using matrix notation, define inductively

$$s^t_{(0)} = s$$

and

$$s^t_{(k+1)} = s + \int_0^t L^u \cdot s^u_{(k)}\, du.$$

As before the $s^t_{(k)}$ converge in the topology of simple convergence in $\mathcal{E}'_p = \mathcal{E}_p \otimes_{\mathcal{O}_p} \mathcal{O}'_p$ to s^t. We first prove:

LEMMA 2. *The following holds in* $\mathcal{E}'_p$ *for* $k \geqslant 0$:

$$s^t_{(k)} \equiv \sum_{i=0}^{k} \frac{1}{i!} (tL)^i \cdot s \mod \mathcal{J}_p . \mathcal{E}'_p$$

(*as before* $\mathcal{J}_p$ *denotes the ideal* $t.\mathcal{J}_p(a)$).

PROOF. This is trivially true for $k = 0$, and if true for $k = N$, then as in Lemma 5.1.7,

$$s^t_{(N+1)} \equiv s + \sum_{i=0}^{N} \int_0^t \frac{1}{i!} L^u (uL)^i \cdot s\, du$$

$$\equiv \sum_{i=0}^{N+1} \frac{1}{i!} (tL)^i \cdot s \mod \mathcal{J}_p . \mathcal{E}'_p$$

as required.

Thus if we write formally

$$e^{tL} = \sum_{i=0}^{\infty} (tL)^i / i!$$

and define an element $\mathrm{ch}(E, z, t) \in \mathcal{O}'_p$ by

$$\mathrm{ch}(E, z, t) = \mathrm{trace}\; e^{tL(z)}$$

we obtain the following theorem:

THEOREM 3. *Let* dz *be the element* $dz_1 \wedge \cdots \wedge dz_n$ *of* Ω^n_p. *Then for* $t \in \mathbf{C} - \{0\}$ *sufficiently small, the following holds in* $\mathrm{Ext}^n_{\mathcal{O}_p}(\mathcal{O}_p / \mathcal{J}_p(a), \Omega^n_p)$:

$$\left[\frac{\mathrm{trace}\; \varphi_t(z)\, dz}{z_1 - z_1^t, \ldots, z_n - z_n^t} \right] = \left[\frac{t^{-n} \times \mathrm{ch}(E, z, t) \times T(z, t)\, dz}{a_1, \ldots, a_n} \right].$$

REMARK. The expression for the fixed-point index obtained by applying the Grothendieck residue to the right-hand side of the equality given by Theorem 2 can also be derived by purely analytic methods, using the integral formula expression for the residue, and a local perturbation of the vector field to obtain nondegenerate zeroes; see [16]. However the generalisation given by the above theorem appears to be less amenable to this approach, except in the case where E is some tensor or exterior power of the holomorphic contangent bundle. This is because it is not clear in general how to extend the perturbation on M to a compatible perturbation of the $\mathbf{C}^+$ action on E, although this may be possible in particular cases.

REMARK. Note that if $\mathcal{E} = \Omega^1$, the sheaf of germs of sections of the holomorphic cotangent bundle of M with the corresponding geometric endomorphism given by the differential of the endomorphism of M, then taking the usual generating sections $dz_1, \ldots, dz_n$,

$$L_{ij} = \partial^2 z_i^t / \partial t \partial z_j = \partial a_i / \partial z_j = A_{ij}.$$

Therefore if we define elements $T^k(z, t) \in \mathcal{O}_p'$ for $0 \leqslant k \leqslant n$:

$$T^k(z, t) = \operatorname{trace} \Lambda^k(e^{tA}) \times \det\big(tA / (I - e^{tA})\big)$$

we can calculate the fixed-point indices for the case $\mathcal{E} = \Omega^k$ by using:

$$\left[\frac{\operatorname{trace} \Lambda^k\big(\partial z_i^t / \partial z_j\big)\, dz}{z_1 - z_1^t, \ldots, z_n - z_n^t} \right] = \left[\frac{t^{-n} \times T^k(z, t)\, dz}{a_1, \ldots, a_n} \right].$$

Of course, this result could also be obtained directly from the work of §4. If (4.2) is differentiated with respect to z it gives

$$\big(\partial z_i^t / \partial z_j\big) = e^{tA} \mod \mathcal{I}_p$$

which immediately implies the above equality.

REMARK. One or two observations are in order concerning the matrix L, which may be regarded as an endomorphism L: $\mathcal{E}_p \to \mathcal{E}_p$ where, for $s \in \mathcal{E}_p$,

$$L(s) = \partial s^t / \partial t|_{t=0}.$$

Note that if E is the trivial line bundle, the corresponding endomorphism of $\mathcal{O}_p$ is simply $f \to X \cdot f$ where $X \cdot f$ is the derivative of f along the vector field, i.e. $\sum_{i=1}^n a_i \partial f / \partial z_i$. Thus

$$L(fs) = (X \cdot f)s + fL(s).$$

This implies that L induces a well-defined linear transformation of the fibre of E at the zero p of X, which with respect to the basis $\{s_i(p)\}$ of the fibre is given by the matrix $L_{ij}(p)$. In case $\mathcal{E} = \Omega^1$, then L is essentially the Lie bracket action and the eigenvalues of the matrix $A_{ij}(0)$ are the *characteristic roots* of the vector field at p.

Compare the situation with that of [4], where closely related results on zeroes of holomorphic vector fields are given.

REMARK. The explanation for the notation T and ch is as follows.

Define elements $c_i \in \mathcal{O}_p$ for $1 \leqslant i \leqslant m$ by

$$\det(I + xL(z)) = 1 + \sum_{i=1}^{m} x^i c_i(z)$$

where x is an indeterminate. In fact the c_i depend on the particular coordinates chosen around p, but it is not difficult to check that their classes mod $\mathcal{G}_p(a)$ are in fact independent of the coordinates. In any case when $L = A$, then $T(z, t)$ is essentially the (dual) Todd class in the 'Chern classes' $c_i(z)$ and when L is the matrix associated to the $\mathbf{C}^+$-vector bundle E, then $\operatorname{ch}(E, z, t)$ is the Chern character in the classes $c_i(z)$ (see [11]).

REMARK. If we let

$$v_p(E, t) = \frac{1}{t^n} \operatorname{Res}_p \left[\frac{\operatorname{ch}(E, z, t) T(z, t)\, dz}{a_1, \ldots, a_n} \right],$$

then the techniques used in [16] yield immediately the following analogue of Theorem 2 of that paper.

THEOREM. *Let λ_i for $1 \leqslant i \leqslant n$ be the characteristic roots of the vector field at p, i.e. the eigenvalues of the matrix $\partial a_i / \partial z_j(p)$, and let $Y_i(t) = (1 - e^{\lambda_i t})^{-1}$ if $\lambda_i \neq 0$, and $Y_i(t) = 0$ otherwise. Similarly, for $1 \leqslant i \leqslant m$, let μ_i be the eigenvalues of the matrix $L(p)$. Then for t sufficiently small and nonzero*:

$$v_p(E, t) = t^{-n} \sum_{i=1}^{m} e^{\mu_i t} Q_{p,i}(t, Y_1(t), \ldots, Y_n(t))$$

for certain polynomials $Q_{p,i}$ in $n + 1$ variables with coefficients in $\mathbf{C}$.

6. **Examples.** By analogy with [11] we may write

$$T(y; z, t) = \sum_{k=0}^{n} T^k(z, t) y^k$$

and

$$\alpha_p(y; t) = \left[\frac{t^{-n} T(y; z, t)\, dz}{a_1, \ldots, a_n} \right]$$

so that

$$\alpha_p(y; t) = \sum_{m=0}^{\infty} \alpha_{p,m}(y) t^{m-n}.$$

With this notation, Theorem 3 of [16] may be written as

$$\sum_p \operatorname{Res}_p(\alpha_{p,n}(y)) = \chi_y(M),$$

$$\sum_p \operatorname{Res}_p(\alpha_{p,m}(y)) = 0 \quad \text{for } m \neq n$$

where $\chi_y(M)$ is the χ_y-genus of M, as defined in [11].

EXAMPLE 1. Setting $y = -1$ it is clear that

$$\alpha_{p,n}(-1) = \left[\frac{da_1 \wedge \cdots \wedge da_n}{a_1, \ldots, a_n} \right]$$

and it follows from properties of the residue [9, III.9.R6] that $\mathrm{Res}_p(\alpha_{p,n}(-1)) = \dim_{\mathbf{C}}(\mathcal{O}_p / \mathcal{I}_p(a))$. But this is the *multiplicity* of the zero of X, and since $\chi_{-1}(M)$ is simply the Euler-Poincaré characteristic, we recover a special case of the classical Hopf theorem.

EXAMPLE 2. In the case where $\dim_{\mathbf{C}} M = 1$, the vector field may be written locally as $a(z)\partial/\partial z$, and

$$\alpha_p(y; t) = \left[a'(1 + y \exp ta')(1 - \exp ta')^{-1} dz/a \right]$$

where $a' = \partial a/\partial z$. In this case the Grothendieck residue coincides with the classical Cauchy residue and if $a'(p) = \lambda \neq 0$ then

$$\mathrm{Res}_p(\alpha_p(y; t)) = (1 + ye^{\lambda t})/(1 - e^{\lambda t})$$

while if $a(z)$ has a zero of order $M > 1$ at p,

$$\mathrm{Res}_p(\alpha_p(y; t)) = -(1 + y)(2\pi i t)^{-1} \oint dz/a + (1 - y)M/2.$$

In dimension n we have the following results. The first can be found in [15] and the second, which was conjectured by G. Lusztig, is proved in [16]. In fact it was the attempt to prove this conjecture that led to most of the results given here and in [16]. We write $A(p)$ for the matrix $\partial a_i/\partial z_j(p)$.

(a) If the zero of X is nondegenerate, i.e. $A(p)$ is nonsingular, it is not difficult to show that $\mathrm{Res}_p(\alpha_p(y; t))$ is a polynomial in y whose coefficients are bounded as $t \to \infty$ radially in all but a finite number of directions.

(b) At the other extreme, if the transformation given by $A(p)$ is nilpotent, then the coefficients of the powers of y in $\mathrm{Res}_p(\alpha_p(y; t))$ are of the form:

$$t^{-n} \times \text{polynomial in } t.$$

One might then ask if the behaviour of (a) occurs in case (b). In other words, are the polynomials in t always of degree $\leqslant n$? The previous example shows that this is certainly the case in dimension 1, but the vector field described below gives a counterexample in dimension 3. However, I know of no counterexample in dimension 2, or in the case of a holomorphic vector field which is defined globally on a compact manifold.

EXAMPLE 3. Let (x, y, z) be coordinates for $\mathbf{C}^3$, and take X to be the vector field

$$X(x, y, z) = a\partial/\partial x + b\partial/\partial y + c\partial/\partial z$$

where

$$a(x,y,z) = x^2 + ye^x,$$
$$b(x,y,z) = y^2 + z,$$
$$c(x,y,z) = z^2.$$

This has an isolated zero at the origin, and $A(0)$ is clearly nilpotent. The algorithm described in [3] will be used to find the coefficient of t in

$$\text{Res}_0(\alpha_0(0; t)).$$

First note that

$$x^8 = (x^6 - x^4ye^x + x^2y^2e^{2x} - y^3e^{3x})a + e^{4x}(y^2 - z)b + e^{4x}c,$$
$$y^4 = (y^2 - z)b + c,$$
$$z^2 = c.$$

The Nullstellensatz for germs of analytic functions ensures the existence of such 'multipliers' and this example has been chosen to make them easy to find. Let $C(x, y, z)$ be the determinant of the matrix of multipliers:

$$C(x,y,z) = (x^6 - x^4ye^x + x^2y^2e^{2x} - y^3e^{3x})(y^2 - z).$$

Setting $a_x = 2x + ye^x$ it is easily checked that

$$T^0(x,y,z,t) \equiv -(1 - a_xt/2 + a_x^2t^2/12 - a_x^4t^4/720)$$
$$\times (1 - yt + y^2t^2/3) \times (1 - zt) \mod(x^8, y^4, z^2, t^5).$$

Then, applying the algorithm described in [3], the required number is the coefficient of $x^7y^3zt^4$ in $C(x, y, z) \times T^0(x, y, z, t)$, and this may be checked to be $-1/90$.

7. **Additive group actions on algebraic varieties.** Analogous results hold for additive group actions on algebraic varieties defince over an algebraically closed field of characteristic zero. This is done by replacing the analytic criteria for convergence used in the preceding work by convergence arguments in the m-adic topology of the local ring at a fixed point. Suppose the additive group A^+ of the algebraically closed field k acts rationally on the smooth variety M, defined over k, by

$$f: M \times A^+ \to M.$$

Assume that $p \in M$ is an isolated fixed point for the action of A^+. Note that it is proved in [13] that if M is complete and connected then the fixed-point set is connected, so that p will be the *only* fixed point. Perhaps the simplest example of this situation is the action of A^+ on $P_1(k) = A^+ \cup \{\infty\}$ given by $(z, t) \to z + t$ with fixed point $\{\infty\}$.

As before let $\mathcal{O}_p$ and $\mathcal{O}'_p$ be the local rings at $p \in M$ and $(p, 0) \in M \times A^+$, with maximal ideals m_p and m'_p respectively. Let $z_1, \ldots, z_n \in \mathcal{O}_p$ be regular parameters for M at p and let $f_i = f^* z_i \in \mathcal{O}_p'$. The corresponding "germ of vector field" can then be defined by $a_i = \partial f_i / \partial t|_{t=0}$. The problem then comes down to constructing a solution for the formal differential equations

$$\partial f_i / \partial t(z, t) = a_i(f(z, t))$$

in the ring $\mathcal{O}_p'$, subject to the initial conditions $f_i(z, 0) = z_i$. It is then necessary to prove uniqueness in order to identify the solution with the given A^+ action. As in the analytic case this can be done by defining an endomorphism α of the m_p'-adic completion $(\mathcal{O}_p')^{\hat{}n}$ of $(\mathcal{O}_p')^n$ by

$$\alpha(g) = z + \int_0^t a(g(z, u))\, du$$

for $g \in m_p'(\mathcal{O}_p')^{\hat{}n}$. (The formal integration needs the hypothesis that k is of zero characteristic.) It is then trivial to prove that α is a "contraction map" for the m_p'-adic topology, and the existence and uniqueness of the solution in $(\mathcal{O}_p')^{\hat{}n}$ follows immediately. However, it is more convenient to prove the following slightly stronger result.

Let $(\mathcal{O}_p')^{\tilde{}}$ be the completion of $\mathcal{O}_p'$ with respect to the m_p-adic topology (this topology is finer than the m_p'-adic topology). Then α defines an endomorphism of $m_p(\mathcal{O}_p')^{\tilde{}n}$ and in order to show that a solution of the differential equations exists in $(\mathcal{O}_p')^{\tilde{}}$ it is only necessary to prove the following proposition.

PROPOSITION 2. *The endomorphism α is a contraction map for the m_p-adic topology.*

PROOF. For $g, h \in m_p(\mathcal{O}_p')^{\tilde{}n}$ with $g - h \in m_p^k(\mathcal{O}_p')^{\tilde{}n}$ for $k \geqslant 1$ we have

$$\alpha(g) - \alpha(h) \equiv A \cdot \int_0^t (g(z, u) - h(z, u))\, du \mod m_p^{2k}(\mathcal{O}_p')^{\tilde{}n}$$

where as before $A_{ij}(z) = \partial a_i / \partial z_j$ in the m_p-adic completion $\mathcal{O}_p^{\hat{}}$ of $\mathcal{O}_p$. Now A^+ has a rational representation on the n-dimensional vector space m_p / m_p^2, say $t \to U_t$. Since A^+ is a unipotent group the matrices U_t are unipotent and $\partial U_t / \partial t|_{t=0} = A(p)$ is nilpotent. Thus for sufficiently large N the matrix A^N has entries in $m_p \mathcal{O}_p^{\hat{}}$ and

$$\alpha^N(g) - \alpha^N(h) \in M_p^{k+1}(\mathcal{O}_p')^{\tilde{}n}$$

as required.

Again we obtain the congruence

$$z - f(z, t) \equiv \tau(z, t) \cdot ta(z) \mod \mathcal{I}_p^2(\mathcal{O}_p')^{\tilde{}n}$$

where as before $\mathcal{I}_p = t.\mathcal{I}_p(a)$. Note that for k large enough, $m_p^k \subset \mathcal{I}_p(a)$, since the zero is isolated. Thus no essential information has been lost by taking the completion.

The generalisation of §5 can be carried through similarly in the algebraic case. Note that in all cases the residue at the fixed point will be of the form $t^{-n} \times$ polynomial in t.

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